GENERALIZED INJECTIVITY OF BANACH MODULES

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ABSTRACT. In this paper, we study the notion of ϕ-injectivity in the special case that $\phi = 0$. For an arbitrary locally compact group G, we characterize the 0-injectivity of $L^1(G)$ as a left $L^1(G)$ module. Also, we show that $L^1(G)^{**}$ and $L^p(G)$ for $1 < p < \infty$ are 0-injective Banach $L^1(G)$ modules.

1. INTRODUCTION

The homological properties of Banach modules such as injectivity, projectivity, and flatness was first introduced and investigated by Helemskii; see [5, 6]. White in [11] gave a quantitative version of these concepts, i.e., he introduced the concepts of C-injective, C-projective, and C-flat Banach modules for a positive real number C. Recently Nasr-Isfahani and Soltani Renani introduce a version of these homological concepts based on a character of a Banach algebra A and they showed that every injective (projective, flat) Banach module is a character injective (character projective, character flat respectively) module but the converse is not valid in general. With use of these new homological concepts, they gave a new characterization of ϕ-amenability of Banach algebra A such that $\phi \in \Delta(A)$ and a necessary condition for ϕ-contractibility of A; see [8].

2. PRELIMINARIES

Let A be a Banach algebra and $\Delta(A)$ denote the character space of A, i.e., the space of all non-zero homomorphisms from A onto $\mathbb{C}$. We denote by **A-mod** and **mod-A** the category of all Banach left A-modules and all Banach right A-modules respectively. In the case that A has an identity we denote by **A-unmod** the category of all Banach left unital modules. For $E, F \in$ **A-mod**, let ${}_AB(E, F)$ be the space of all bounded linear left A-module morphisms from E into F.

For each Banach space E, $B(A, E)$; the Banach algebra consisting of all bounded linear operator from A into E, is in **A-mod** with the following module action:

$$(a \cdot T)(b) = T(ba) \quad (T \in B(A, E), a, b \in A).$$

Definition 2.1. Let A be a Banach algebra and $J \in$ **A-mod**. We say that J is injective if for each $F, E \in$ **A-mod** and admissible monomorphism $T : F \to E$ the induced map $T_J : {}_AB(E, J) \to {}_AB(F, J)$ defined by $T_J(R) = R \circ T$ is onto.

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Suppose that $\phi \in \Delta(A)$. For $E \in$ **A-mod**, put

$$I(\phi, E) = \mathrm{span}\{a \cdot \xi - \phi(\xi)a : a \in A, \xi \in E\},$$
$$_{\phi}B(A^{\sharp}, E) = \{T \in B(A^{\sharp}, E) : T(ab - \phi(b)a) = a \cdot T(b - \phi(b)e^{\sharp}), \quad (a, b \in A)\}.$$

It is clear that $I(\phi, E) = \{0\}$ if and only if the module action of E is given by $a \cdot x = \phi(a)x$ for all $a \in A$ and $x \in E$.

Obviously, $_{\phi}B(A^{\sharp}, E)$ is a Banach subspace of $B(A^{\sharp}, E)$. On the other hand, for each $b \in \ker(\phi)$, if $T \in {}_{\phi}B(A^{\sharp}, E)$, then $T(ab) = a \cdot T(b)$ for all $a \in A$. Therefore, we conclude that $_{\phi}B(A^{\sharp}, E)$ is a Banach left A-submodule of $B(A^{\sharp}, E)$.

Note that if $E, F \in$ **A-mod** and $\rho : E \to F$ is a left A-module homomorphism, we can extend the module actions of E and F from A into $A^{\sharp}$ and ρ to a left $A^{\sharp}$-module homomorphism in the following way:

$$(a, \lambda) \cdot e = a \cdot e + \lambda e \quad (a \in A, \lambda \in \mathbb{C}, e \in E)$$
$$(a, \lambda) \cdot f = a \cdot f + \lambda f \quad (a \in A, \lambda \in \mathbb{C}, f \in F).$$

So, $\rho((a, \lambda) \cdot e) = a \cdot \rho(e) + \lambda\rho(e) = (a, \lambda) \cdot \rho(e)$.

For Banach spaces E and F, $T \in B(E, F)$ is admissible if and only if there exists $S \in B(F, E)$ such that $T \circ S \circ T = T$.

The following definition of a *ϕ-injective* Banach module, introduced by Nasr-Isfahani and Soltani Renani in [8].

Definition 2.2. Let A be a Banach algebra, $\phi \in \Delta(A)$ and $J \in$ **A-mod**. We say that J is ϕ-injective if for each $F, E \in$ **A-mod** and admissible monomorphism $T : F \to E$ with $I(\phi, E) \subseteq \mathrm{Im}T$, the induced map T_J is onto.

By Definition 2.1 and 2.2, one can easily check that each injective module is ϕ-injective, although by [8, Example 2.5], the converse is not valid. In [4], the authors with use of the semigroup algebras, gave two good examples of ϕ-injective Banach modules which they are not injective.

Let E, F be in **A-mod**. An operator $T \in {}_{A}B(E, F)$ is called a *retraction* if there exists an $S \in {}_{A}B(F, E)$ such that $T \circ S = Id_F$. In this case F is called a retract of E. Also, an operator $T \in {}_{A}B(E, F)$ is called a *coretraction* if there exists an $S \in {}_{A}B(F, E)$ such that $S \circ T = Id_E$.

For $E \in$ **A-mod**, let $_{\phi}\Pi^{\sharp} : E \to {}_{\phi}B(A^{\sharp}, E)$ be defined by $_{\phi}\Pi^{\sharp}(x)(a) = a \cdot x$ for all $a \in A^{\sharp}$ and $x \in E$.

Theorem 2.3. [8, Theorem 2.4] *Let A be a Banach algebra and $\phi \in \Delta(A)$. For $J \in$ **A-mod** the following statements are equivalent.*

(1) *J is ϕ-injective.*
(2) *$_{\phi}\Pi^{\sharp} \in {}_{A}B(J, {}_{\phi}B(A^{\sharp}, J))$ is a coretraction.*

3. 0-injectivity of Banach modules

In this section, we give the definition of a *0-injective* Banach left A-module and show that this class of Banach modules are strictly larger than the class of injective Banach modules.

For each $E \in \mathbf{A\text{-}mod}$ define

$$ {}_0B(A^\sharp, E) = \{T \in B(A^\sharp, E) : T(ab) = a \cdot T(b) \text{ for all } a, b \in A\}. $$

Clearly, ${}_0B(A^\sharp, E)$ is a Banach left A-submodule of $B(A^\sharp, E)$. It is well-known that E^* is in **mod-A** with the following module action:

$$ (f \cdot a)(x) = f(a \cdot x) \quad (a \in A, x \in E, f \in E^*). $$

Definition 3.1. Let A be a Banach algebra and $E \in \mathbf{A\text{-}mod}$. We say that E is (left) 0-injective if for each $F, K \in \mathbf{A\text{-}mod}$ and admissible monomorphism $T : F \to K$ for which $A \cdot K = \text{span}\{a \cdot k : a \in A, k \in K\} \subseteq \text{Im}T$, the induced map T_J is onto.

Similarly, one can define the concept of (right) 0-injective A-module. We say that $E \in \mathbf{A\text{-}mod}$ is 0-flat if $E^* \in \mathbf{mod\text{-}A}$ is (right) 0-injective.

Clearly, each injective module is 0-injective.

We use of the following characterization of 0-injectivity in the sequel without giving the reference.

Proposition 3.2. *Let A be a Banach algebra and $E \in \mathbf{A\text{-}mod}$. Then E is 0-injective if and only if ${}_0\Pi^\sharp$ is a coretraction.*

Proof. Suppose $E \in \mathbf{A\text{-}mod}$ is 0-injective. Take $F = E$, $K = {}_0B(A^\sharp, E)$ and $T = {}_0\Pi$. Then $A \cdot K \subseteq \text{Im}({}_0\Pi)$ and $a \cdot T = {}_0\Pi(T(a))$ for each $a \in A$ and $T \in K$. Hence, for the identity map $I_E \in_A B(F, E) =_A B(E, E)$, there exists $\rho \in_A B(K, E) =_A B({}_0B(A^\sharp, E), E)$ such that $\rho \circ {}_0\Pi = \rho \circ T = I_E$.

Conversely, let ${}_0\rho : {}_0B(A^\sharp, E) \longrightarrow E$ be a left A-module morphism and a left inverse for the canonical morphism ${}_0\Pi$. Suppose that $F, K \in \mathbf{A\text{-}mod}$ and $T : F \longrightarrow K$ is an admissible monomorphism such that $A \cdot K \subseteq \text{Im}T$. Let $W \in_A B(F, E)$ and define the map $R : K \longrightarrow {}_0B(A^\sharp, E)$ by

$$ R(k)(a) = W \circ T'(a \cdot k) \quad (k \in K, a \in A^\sharp), $$

where $T' \in B(K, F)$ satisfies $T \circ T' \circ T = T$. We show that R is well defined, i.e., $R(k) \in {}_0B(A^\sharp, E)$ for each $k \in K$. So, we will show that $R(k)(ab) = a \cdot R(k)(b)$ for each $a, b \in A$. By assumption $A \cdot K \subseteq \text{Im}T$ and so there exist $f \in F$ such that $b \cdot k = T(f)$. Therefore

$$ \begin{aligned} a \cdot R(k)(b) &= a \cdot W \circ T'(b \cdot k) = a \cdot W \circ T'(T(f)) \\ &= a \cdot W(f) = W(a \cdot f) \\ &= W \circ T'(T(a \cdot f)) = W \circ T'(ab \cdot k) \\ &= R(k)(ab). \end{aligned} $$

Moreover, for each $b \in A^\sharp$ we have

$$ \begin{aligned} R(a \cdot k)(b) &= W \circ T'(b \cdot (a \cdot k)) = W \circ T'(ba \cdot k) \\ &= R(k)(ba) = (a \cdot R(K))(b). \end{aligned} $$

It follows that $R(a \cdot k) = a \cdot R(k)$. Now, take $S = {}_0\rho \circ R \in {}_AB(K, E)$. Since $R \circ T = {}_0\Pi \circ W$, we conclude that $S \circ T = W$, which completes the proof. $\square$

Now, we give a sufficient condition for 0-inectivity which provide for us a large class of Banach algebras A such that they are 0-injective in **A-mod**.

Recall that by [10, Corollary 2.2.8(i)], if $A \in$ **A-mod** is injective, then A has a right identity. Moreover, the converse is not valid in general even in the case that A has an identity; see Example 3.4.

Proposition 3.3. *Let A be a Banach algebra. If A has an identity, then $A \in$* **A-mod** *is 0-injective.*

Proof. Let e be the identity of A. Define $\rho :_0B(A^\sharp, A) \to A$ by $\rho(T) = T(e)$ for all $T \in {}_0B(A^\sharp, A)$. It is obvious that ρ is a left inverse for ${}_0\Pi^\sharp$, because for each $a \in A$, we have

$$\rho \circ {}_0\Pi^\sharp(a) = ({}_0\Pi^\sharp(a))(e) = ea = a.$$

Also, ρ is a left A-module morphism, because for each $a \in A$ and $T \in {}_0B(A^\sharp, A)$ we have

$$\begin{aligned} \rho(a \cdot T) &= (a \cdot T)(e) = T(ea) = T(a) \\ a \cdot \rho(T) &= a \cdot T(e) = T(ae) = T(a). \end{aligned}$$

Therefore, $A \in$ **A-mod** is 0-injective. $\square$

For each locally compact group G, let $M(G)$ be the Banach algebra consisting of all complex regular Borel measure of G and let $L^\infty(G)$ be the space of all measurable complex-valued functions on G which they are essentially bounded; see [1] for more details.

The group G is said to be *amenable* if there exists an $m \in L^\infty(G)^*$ such that $m \geq 0$, $m(1) = 1$ and $m(L_x f) = m(f)$ for each $x \in G$ and $f \in L^\infty(G)$, where $L_x f(y) = f(x^{-1}y)$.

As an application of the above theorem we give the following example which shows the difference between 0-injectivity and injectivity.

Example 3.4. Let G be a non-amenable locally compact group. Then by [10, Theorem 3.1.2], $M(G) \in$ **M(G)-mod** is not injective, but it is 0-injective.

By [6, Proposition VII.1.35], if $E \in$ **A-unmod**, each retract of E is injective. For 0-injective Banach modules we have the following proposition.

Proposition 3.5. *Let A be a Banach algebra and let $E \in$* **A-mod** *be 0-injective. Then each retract of E is also 0-injective.*

Proof. Let $F \in$ **A-mod** be a retract of E. Also, let $T \in {}_AB(E, F)$ and $S \in {}_AB(F, E)$ be such that $T \circ S = I_F$.

Since $E \in$ **A-mod** is 0-injective, there exists ${}_E\rho^\sharp \in {}_AB({}_0B(A^\sharp, E), E)$ for which ${}_E\rho^\sharp \circ {}_E\Pi^\sharp(x) = x$ for all $x \in E$.

Now, define the map ${}_F\rho^\sharp : {}_0B(A^\sharp, F) \to F$ by

$${}_F\rho^\sharp(W) = T \circ {}_E\rho^\sharp(S \circ W) \quad (W \in {}_0B(A^\sharp, F)).$$

It is straightforward to check that ${}_F\rho^\sharp$ is a left A-module morphism. On the other hand, for each $y \in F$ we have

$$\begin{aligned} {}_F\rho^\sharp \circ {}_F\Pi^\sharp(y) &= {}_F\rho^\sharp({}_F\Pi^\sharp(y)) \\ &= T \circ {}_E\rho^\sharp(S \circ {}_F\Pi^\sharp(y)) \\ &= T \circ {}_E\rho^\sharp({}_E\Pi^\sharp(S(y))) \\ &= T \circ S(y) = y. \end{aligned}$$

Therefore, $F \in$ **A-mod** is 0-injective. □

Now, we try to characterize the 0-injectivity of $L^1(G)$ in $\mathbf{L^1(G)}$**-mod**. First we give the following lemma.

Lemma 3.6. *Let A be a Banach algebra and $E \in$ **A-mod**. If E is 0-injective, then*

$${}_0B(A^\sharp, E) = \{T : T = R_x \text{ on } A \text{ for some } x \in E\},$$

where $R_x a = a \cdot x$ for all $a \in A$.

Proof. Let $E \in$ **A-mod** be 0-injective. So, there exists ${}_0\rho^\sharp \in {}_AB({}_0B(A^\sharp, E), E)$ with ${}_0\rho^\sharp \circ {}_0\Pi^\sharp(x) = x$ for all $x \in E$.

Let T be an element of ${}_0B(A^\sharp, E)$. Hence

$$\begin{aligned} T(b) = {}_0\rho^\sharp \circ {}_0\Pi^\sharp(T(b)) &= {}_0\rho^\sharp({}_0\Pi^\sharp(T(b))) \\ &= {}_0\rho^\sharp(b \cdot T) \\ &= b \cdot {}_0\rho^\sharp(T). \end{aligned}$$

Take $x_0 = {}_0\rho^\sharp(T)$. So, $T = R_{x_0}$ on A and this completes the proof. □

Recall that $E \in$ **A-mod** is faithful in A, if for each $x \in E$, the relation $a \cdot x = 0$ for all $a \in A$, implies $x = 0$.

Theorem 3.7. *Let G be a locally compact group. Then $L^1(G) \in \mathbf{L^1(G)}$**-mod** is 0-injective if and only if G is discrete.*

Proof. Let G be a discrete group. Then $L^1(G)$ is unital and so the result follows from Proposition 3.3.

Conversely, let G be non-discrete. So, $L^1(G) \neq M(G)$. Suppose that $\mu \in M(G) \setminus L^1(G)$. Since $L^1(G)$ is an ideal of $M(G)$, the operator T_μ defined by

$$T_\mu((f, \lambda)) = f \cdot \mu \quad ((f, \lambda) \in L^1(G)^\sharp),$$

is in ${}_0B(L^1(G)^\sharp, L^1(G))$, but it is not of the form R_x for some $x \in L^1(G)$, because $M(G)$ is faithful in $L^1(G)$. Therefore, by Lemma 3.6, $L^1(G)$ in $\mathbf{L^1(G)}$**-mod** is not 0-injective. □

Recall that a Banach algebra A is left 0-amenable if for every Banach A-bimodule X with $a \cdot x = 0$ for all $a \in A$ and $x \in X$, every continuous derivation $D : A \to X^*$ is inner, or equivalently, $H^1(A, X^*) = 0$ where $H^1(A, X^*)$ denotes the first cohomology group of A with coefficients in X^*; see [7] for more details.

Now, we investigate the relation between 0-injectivity and 0-amenability.

Let $E, F \in$ **A-mod**. Suppose that $Z^1(A \times E, F)$ denotes the Banach space of all continuous bilinear maps $B : A \times E \longrightarrow F$ satisfying

$$a \cdot B(b, \xi) - B(ab, \xi) + B(a, b \cdot \xi) = 0 \quad (a, b \in A, \xi \in E).$$

Define $\delta_0 : B(E, F) \longrightarrow Z^1(A \times E, F)$ by $(\delta_0 T)(a, \xi) = a \cdot T(\xi) - T(a \cdot \xi)$ for all $a \in A$ and $\xi \in E$. Then we have

$$\mathrm{Ext}^1_A(E, F) = Z^1(A \times E, F)/\mathrm{Im}\delta_0.$$

By [5, Proposition VII.3.19], we know that $\mathrm{Ext}^1_A(E, F)$ is topologically isomorphic to $H^1(A, B(E, F))$ where $B(E, F)$ is a Banach A-bimodule with the following module actions:

$$(a \cdot T)(\xi) = a \cdot T(\xi), \quad (T \cdot a)(\xi) = T(a \cdot \xi) \quad (a \in A, \xi \in E, T \in B(E, F)).$$

To see further details about $\mathrm{Ext}^1_A(E, F)$; see [6].

Lemma 3.8. *Let $E \in$ **A-mod**. If $\mathrm{Ext}^1_A(F, E) = \{0\}$ for all $F \in$ **A-mod** with $A \cdot F = 0$, then $E \in$ **A-mod** is 0-injective.*

Proof. To show this, let $K, W \in$ **A-mod** and $T : K \to W$ be an admissible monomorphism with $A \cdot W \subseteq \mathrm{Im}T$. We claim that the induced map T_E is onto.

We know that the short complex $0 \to K \xrightarrow{T} W \xrightarrow{q} \frac{W}{\mathrm{Im}T} \to 0$ is admissible where q is the quotient map. But for all $a \in A$ and $x \in W$, $a \cdot (x + \mathrm{Im}T) = \mathrm{Im}T$, because $A \cdot W \subseteq \mathrm{Im}T$. Therefore, by assumption $\mathrm{Ext}^1_A(\frac{W}{\mathrm{Im}T}, E) = \{0\}$. Now, by [6, III Theorem 4.4], the complex

$$0 \to \ _AB(\frac{W}{\mathrm{Im}T}, E) \to \ _AB(W, E) \xrightarrow{T_E} \ _AB(K, E) \to \mathrm{Ext}^1_A(\frac{W}{\mathrm{Im}T}, E) \to \cdots,$$

is exact. Therefore, T_E is onto.

□

Recall that if E, F be two Banach spaces and $E\widehat{\otimes}F$ denotes the projective tensor product space, then $(E\widehat{\otimes}F)^*$ is isomorphic to $B(E, F^*)$ as two Banach spaces with the pairing

$$< Tx, y >= T(x \otimes y) \quad (x \in E, y \in F, T \in (E\widehat{\otimes}F)^*).$$

Also, note that $E\widehat{\otimes}F$ is isometrically isomorphic to $F\widehat{\otimes}E$ as two Banach spaces.

Theorem 3.9. *Let A be a Banach algebra. Then A is left 0-amenable if and only if each $J \in$ **mod-A** is 0-flat.*

Proof. Suppose that A is left 0-amenable. We show that $\mathrm{Ext}^1_A(E, J^*) = \{0\}$ for all $E \in$ **A-mod** with $A \cdot E = 0$. We have

$$\mathrm{Ext}^1_A(E, J^*) = H^1(A, B(E, J^*)) = H^1(A, (E\widehat{\otimes}J)^*) = \{0\},$$

because $E\widehat{\otimes}J \in$ **mod-A** has the module action, $a \cdot z = 0$ for all $z \in E\widehat{\otimes}J$. Therefore, by Lemma 3.8, $J^* \in$ **A-mod** is 0-injective.

Conversely, let $J \in$ **mod-A** be 0-flat. So, for Banach right A-module $\mathbb{C}$ with module action $\lambda \cdot a = 0$ for all $a \in A$ and $\lambda \in \mathbb{C}$ we have

$$\begin{aligned} H^1(A, J^*) = H^1(A, B(J, \mathbb{C})) &= H^1(A, B(J, \mathbb{C}^*)) \\ &= H^1(A, (J \widehat{\otimes} \mathbb{C})^*) \\ &= H^1(A, (\mathbb{C} \widehat{\otimes} J)^*) \\ &= H^1(A, B(C, J^*)) \\ &= \mathrm{Ext}^1_A(\mathbb{C}, J^*) \\ &= 0. \end{aligned}$$

Hence, if we take J a left A module with module action $a \cdot x = 0$ for all $a \in A$ and $x \in J$, then the above relation implies that A is 0-amenable. $\square$

By [2, Corollary 4.7], we know that $L^1(G)^{**} \in \mathbf{L^1(G)}$**-mod** is injective if and only if G is an amenable group. Also, if $1 < p < \infty$ by [3, Theorem 9.6], $L^p(G) \in \mathbf{L^1(G)}$**-mod** is injective if and only if G is an amenable group.

Corollary 3.10. *Let G be a locally compact group, $1 < p < \infty$ and $E \in \mathbf{L^1(G)}$-**mod** be $L^p(G)$ or $L^1(G)^{**}$. Then $E \in \mathbf{L^1(G)}$-**mod** is 0-injective.*

Proof. Since $L^1(G)$ has a bounded approximate identity by [7, Proposition 3.4 (i)], we know that $L^1(G)$ is 0-amenable. So, by Theorem 3.9 we conclude the result. The second part follows similarly, because for each $1 < p < \infty$ we know that $L^q(G)^* = L^p(G)$ where q satisfies the relation $q^{-1} + p^{-1} = 1$. $\square$

Remark 3.11. In general, by [7, Proposition 3.4 (i)], if A is a Banach algebra with a bounded approximate identity, then each $E \in$ **mod-A** is 0-flat.

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